Solution to the Sample Test for SBL

1. When $12^x = 18^y = 24^z = 2$, find $\dfrac{1}{x} - \dfrac{1}{y} + \dfrac{1}{z}$.

(Sol) Since $12^x = 18^y = 24^z = 2$, we get $x = \dfrac{\log 2}{\log 12}$, $y = \dfrac{\log 2}{\log 18}$ and $z = \dfrac{\log 2}{\log 24}$, which yields

$$\frac{1}{x} - \frac{1}{y} + \frac{1}{z} = \frac{\log 12}{\log 2} - \frac{\log 18}{\log 2} + \frac{\log 24}{\log 2} = \frac{\log 12 - \log 18 + \log 24}{\log 2}.$$

Since $\log 12 - \log 18 + \log 24 = \log \dfrac{12 \times 24}{18} = \log 16 = 4 \log 2$, it follows that $\dfrac{1}{x} - \dfrac{1}{y} + \dfrac{1}{z} = 4$. The answer is ④.

2. Find $\sin \dfrac{\pi}{12} + \cos \dfrac{\pi}{12}$

(Sol) We note that

$$\left(\sin \frac{\pi}{12} + \cos \frac{\pi}{12} \right)^2 = \sin^2 \frac{\pi}{12} + \cos^2 \frac{\pi}{12} + 2 \sin \frac{\pi}{12} \cos \frac{\pi}{12} = 1 + \sin \frac{\pi}{6}$$
$$= 1 + \frac{1}{2} = \frac{3}{2} .$$

Hence, it follows that $\sin \dfrac{\pi}{12} + \cos \dfrac{\pi}{12} = \sqrt{\dfrac{3}{2}}$.

The answer is ③.

3. Let $A = \begin{pmatrix} 3 & 2 \\ -1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & -3 \\ 4 & 5 \end{pmatrix}$. When $(A^{-1} + B)^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, find $a + b + c + d$.

(Sol) Since $A^{-1} = -\begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & -3 \end{pmatrix}$, it follows that

$A^{-1} + B = \begin{pmatrix} 1 & 2 \\ -1 & -3 \end{pmatrix} + \begin{pmatrix} -3 & -3 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 3 & 2 \end{pmatrix}$. Hence,

$(A^{-1} + B)^2 = \begin{pmatrix} -2 & -1 \\ 3 & 4 \end{pmatrix}^2 = \begin{pmatrix} -2 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} -2 & -1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

which shows that $a + b + c + d = 2$. The answer is ②.

4. When $\omega = \dfrac{\sqrt{3}+i}{\sqrt{2}+\sqrt{2}\,i}$, find ω^{12}.

(Sol) We note that

$\omega^2 = \dfrac{2+2\sqrt{3}i}{4i} = \dfrac{1+\sqrt{3}i}{2i}$, $\omega^4 = \dfrac{-2+2\sqrt{3}i}{-4} = \dfrac{1-\sqrt{3}i}{2}$, and

$\omega^8 = \dfrac{-2-2\sqrt{3}i}{4} = \dfrac{-1-\sqrt{3}i}{2}$. Hence, it follows that

$\omega^{12} = \omega^4 \cdot \omega^8 = \dfrac{1-\sqrt{3}i}{2} \cdot \dfrac{-1-\sqrt{3}i}{2} = \dfrac{-1-3}{4} = -1$.

The answer is ①.

5. Let M and m be the maximum and minimum values of $y = 4x^3 - 3x^2 - 6x + 2$, $(-1 \le x \le 1)$, respectively. Find $M + m$.

(Sol) Since $y' = 12x^2 - 6x - 6 = 6(2x^2 - x - 1) = 6(2x+1)(x-1)$, it follows that $y' = 0$ at $x = -\dfrac{1}{2}$ and $x = 1$. Since $-1 \le x \le 1$, simple computation shows that $y_{x=-1} = 1$, $y_{x=-\frac{1}{2}} = \dfrac{15}{4}$ and $y_{x=1} = -3$. Hence, $M = \dfrac{15}{4}$ and $m = -3$ and $M + m = \dfrac{3}{4}$. The answer is ②.

6. Find $\displaystyle\int_1^2 x\,(x^2-3)^3\,dx$.

(Sol) Setting $u = x^2 - 3$, then $x\,dx = \dfrac{1}{2}du$. If $x = 1$, then $u = -2$, and if $x = 2$, then $u = 1$. Hence,

$$\int_1^2 x\,(x^2-3)^3\,dx = \int_{-2}^1 \frac{1}{2}u^3\,du = \frac{1}{2}\left[\frac{1}{4}u^4\right]_{-2}^1 = \frac{1}{8}(1-16) = -\frac{15}{8} .$$

The answer is ⑤.