Solution to the Sample Test for SOCIE

1. When $\sin\theta\cos\theta = \frac{1}{3}$ for $0 < \theta < \frac{\pi}{4}$, find the value of $\sin\theta - \cos\theta$.

① $-\frac{\sqrt{5}}{3}$ ② $-\frac{2}{3}$ ③ $-\frac{\sqrt{3}}{3}$ ④ $-\frac{\sqrt{2}}{3}$ ⑤ $-\frac{1}{3}$

(Sol) Since $(\sin\theta - \cos\theta)^2 = 1 - 2\sin\theta\cos\theta = \frac{1}{3}$, it follows that $\sin\theta - \cos\theta = \pm\frac{\sqrt{3}}{3}$. Since $0 < \theta < \frac{\pi}{4}$, we have $\sin\theta - \cos\theta < 0$. Hence, $\sin\theta - \cos\theta = -\frac{\sqrt{3}}{3}$.

The answer is ③.

2. Compute $\log_{10}\frac{2}{1} + \log_{10}\frac{3}{2} + \log_{10}\frac{4}{3} + \cdots + \log_{10}\frac{25}{24}$.

① $\log_{10}25$ ② $\log_{10}35$ ③ $\log_{10}45$ ④ $\log_{10}55$ ⑤ $\log_{10}65$

(Sol) $\log_{10}\frac{2}{1} + \log_{10}\frac{3}{2} + \log_{10}\frac{4}{3} + \cdots + \log_{10}\frac{25}{24} = \log_{10}\left(\frac{2}{1}\frac{3}{2}\frac{4}{3}\cdots\frac{25}{24}\right) = \log_{10}25$.

The answer is ①.

3. For $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, we write $ABA^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Find $a+b+c+d$.

① 2 ② 4 ③ 6 ④ 8 ⑤ 10

(Sol) For $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$. Thus $ABA^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -3 & 8 \\ -2 & 5 \end{pmatrix}$, and $a+b+c+d = 8$.

The answer is ④.

4. Evaluate $\lim_{\theta\to 0}\frac{1-\cos 2\theta}{(tg\,(3\theta))^2}$, where $tgA = \frac{\sin A}{\cos A}$.

① $\frac{1}{9}$ ② $\frac{2}{9}$ ③ $\frac{1}{3}$ ④ $\frac{4}{9}$ ⑤ $\frac{5}{9}$

(Sol) We note that

$$\frac{1-\cos 2\theta}{(tg(3\theta))^2} = (1-\cos 2\theta)\cdot\frac{\cos^2 3\theta}{\sin^2 3\theta} = \frac{(1-\cos 2\theta)(1+\cos 2\theta)}{(1+\cos 2\theta)}\frac{\cos^2 3\theta}{\sin^2 3\theta} = \frac{1-\cos^2 2\theta}{(1+\cos 2\theta)}\frac{\cos^2 3\theta}{\sin^2 3\theta}$$

$$= \frac{\sin^2 2\theta \cos^2 3\theta}{(1+\cos 2\theta)\sin^2 3\theta} = \frac{(\sin 2\theta)^2}{(2\theta)^2} \cdot \frac{4}{9} \cdot \frac{(3\theta)^2}{(\sin 3\theta)^2} \cdot \frac{\cos^2 3\theta}{1+\cos 2\theta},$$

which yields $\lim_{\theta \to 0} \frac{1-\cos 2\theta}{\tan^2 3\theta} = \lim_{\theta \to 0} \frac{(\sin 2\theta)^2}{(2\theta)^2} \frac{4}{9} \frac{(3\theta)^2}{(\sin 3\theta)^2} \frac{\cos^2 3\theta}{1+\cos 2\theta} = 1 \cdot \frac{4}{9} \cdot 1 \cdot \frac{1}{2} = \frac{2}{9}$.

The answer is ②.

5. Find the minimum value of $f(x) = 3x^4 - 2x^3 + 6x^2 - 6x + 2$.

① $\frac{1}{16}$ ② $\frac{3}{16}$ ③ $\frac{5}{16}$ ④ $\frac{7}{16}$ ⑤ $\frac{9}{16}$

(Sol) Since $f'(x) = 12x^3 - 6x^2 + 12x - 6 = 6(2x-1)(x^2+1)$, the minimum value is $f\left(\frac{1}{2}\right) = \frac{7}{16}$.

The answer is ④.

6. Find the area of the region enclosed by $y = x^4 - 3x^3 + x^2 - 1$ and $y = x^4 - 3x^3 + x + 1$.

① $\frac{1}{2}$ ② $\frac{3}{2}$ ③ $\frac{5}{2}$ ④ $\frac{7}{2}$ ⑤ $\frac{9}{2}$

(Sol) Since $(x^4 - 3x^3 + x + 1) - (x^4 - 3x^3 + x^2 - 1) = -(x^2 - x - 2) = -(x+1)(x-2)$, two graphs meet at $x = -1$ and $x = 2$. Since $x^4 - 3x^3 + x + 1 \geq x^4 - 3x^3 + x^2 - 1$ on the interval $[-1, 2]$, the area is given by

$$\int_{-1}^{2} \left((x^4 - 3x^3 + x + 1) - (x^4 - 3x^3 + x^2 - 1) \right) dx = \int_{-1}^{2} -(x^2 - x - 2)\, dx$$

$$= \left[-\frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x \right]_{-1}^{2} = \left(-\frac{8}{3} + 2 + 4 \right) - \left(\frac{1}{3} + \frac{1}{2} - 2 \right) = \frac{9}{2}.$$

The answer is ⑤.