Solution to the Sample Test for SOCIE (Scholarship)

1. When $\sin\theta\cos\theta = \frac{1}{3}$, find the value of $\cos 4\theta$.

① $\frac{1}{9}$ ② $\frac{2}{9}$ ③ $\frac{1}{3}$ ④ $\frac{4}{9}$ ⑤ $\frac{5}{9}$

(Sol) Since $\sin 2\theta = 2\sin\theta\cos\theta = \frac{2}{3}$, it follows that $\cos 4\theta = 1 - 2\sin^2 2\theta = \frac{1}{9}$.

The answer is ①.

2. Compute $\sum_{n=1}^{99} \log_{10}\left(1 - \frac{1}{n+1}\right)$.

① -4 ② -2 ③ 0 ④ 2 ⑤ 4

(Sol) $\sum_{n=1}^{99} \log_{10}\left(1 - \frac{1}{n+1}\right) = \sum_{n=1}^{99} \log_{10}\left(\frac{n}{n+1}\right) = \log_{10}\left(\frac{1}{2}\cdot\frac{2}{3}\cdot\cdots\cdot\frac{99}{100}\right) = \log_{10}\frac{1}{100} = -2$.

The answer is ②.

3. When $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $(A^{2022})^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, find $a+b+c+d$.

① -4048 ② -4046 ③ -4044 ④ -4042 ⑤ -4040

(Sol) For $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $A^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$. Simple computation show that $A^2 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$ and $A^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$. In particular, $A^{2022} = \begin{pmatrix} 1 & 4044 \\ 0 & 1 \end{pmatrix}$, and hence $(A^{2022})^{-1} = \begin{pmatrix} 1 & -4044 \\ 0 & 1 \end{pmatrix}$, which shows $a+b+c+d = -4042$.

The answer is ④.

4. ω $x^2 - x + 1 = 0$ $S = \sum_{n=1}^{101} \omega^n$

① $-i$ ② -1 ③ 0 ④ 1 ⑤ i

$\omega^2 - \omega + 1 = 0$ $\omega^3 = \omega^2 - \omega = -1$

$\omega^{102} = (\omega^3)^{34} = (-1)^{34} = 1$ $S = \frac{\omega - \omega^{102}}{1-\omega} = \frac{\omega - 1}{1-\omega} = -1$

The answer is ②.

5. Find the shortest distance between the curve $y=\sqrt{x}$, $(x \geq 0)$ and the point $(1,0)$.

① $\frac{1}{2}$ ② $\frac{\sqrt{2}}{2}$ ③ $\frac{\sqrt{3}}{2}$ ④ 1 ⑤ $\frac{3}{2}$

(Sol) Since any point on the curve $y=\sqrt{x}$ is expressed by $(x, \sqrt{x})$, the distance $\ell(x)$ from $(x, \sqrt{x})$ to $(1,0)$ is given by $\ell(x) = \sqrt{(x-1)^2+(\sqrt{x}-0)^2} = \sqrt{x^2-x+1}$. Since $\ell'(x)=\dfrac{2x-1}{2\sqrt{x^2-x+1}}$, $\ell(x)$ has the minimum value $\frac{\sqrt{3}}{2}$ at $x=\frac{1}{2}$.

The answer is ③.

6. Find the volume of the solid obtained by rotating the region enclosed by $y=-x^2+4$ and $y=x+2$ about the x-axis.

① $\frac{62}{5}\pi$ ② $\frac{74}{5}\pi$ ③ $\frac{86}{5}\pi$ ④ $\frac{98}{5}\pi$ ⑤ $\frac{108}{5}\pi$

(Sol) From $-x^2+4 = x+2$, it follows that $x^2+x-2 = (x+2)(x-1) = 0$, which shows that two graphs meet at $x = -2$ and $x = 1$. Since $x+2 \leq -x^2+4$ on the interval $[-2,1]$, the volume V is given by

$$
\begin{aligned}
V &= \pi\int_{-2}^{1}\left((-x^2+4)^2-(x+2)^2\right)dx = \pi\int_{-2}^{1}\left(x^4-9x^2-4x+12\right)dx \\
&= \pi\left[\frac{1}{5}x^5-3x^3-2x^2+12x\right]_{-2}^{1} = \pi\left((\frac{1}{5}-3-2+12)-(\frac{-32}{5}+24-8-24)\right) \\
&= \frac{108}{5}\pi .
\end{aligned}
$$

The answer is ⑤.