2021 IUT Admission Test (SBL)

(1) When $\alpha^2 = \sqrt{3+2\sqrt{2}}$, find $\dfrac{\alpha^3 - \alpha^{-3}}{\alpha - \alpha^{-1}}$.

(SOL) : Since $\alpha^2 = \sqrt{3+2\sqrt{2}} = \sqrt{(\sqrt{2}+1)^2} = \sqrt{2}+1$, it follows that $\alpha^{-2} = \dfrac{1}{\sqrt{2}+1} = \sqrt{2}-1$ and $\alpha^2 + \alpha^{-2} = 2\sqrt{2}$.

Hence, $\dfrac{\alpha^3 - \alpha^{-3}}{\alpha - \alpha^{-1}} = \dfrac{(\alpha-\alpha^{-1})(\alpha^2+\alpha\alpha^{-1}+\alpha^{-2})}{(\alpha-\alpha^{-1})} = \alpha^2+1+\alpha^{-2} = 2\sqrt{2}+1.$

(2) Evaluate $\log_3\dfrac{243}{2} - \log_9\dfrac{81}{4}$.

(SOL) : $\log_3\dfrac{243}{2} - \log_9\dfrac{81}{4} = \log_3\dfrac{243}{2} - \dfrac{1}{2}\log_3\left(\dfrac{9}{2}\right)^2 = \log_3\dfrac{243}{2} - \log_3\dfrac{9}{2}$

$= \log_3\left(\dfrac{243}{2}\cdot\dfrac{2}{9}\right) = \log_3 27 = \log_3 3^3 = 3$.

(3) When $a = \dfrac{\sqrt{3}+i}{2}$, find a^{100} .

(SOL) : Since $a^2 = \dfrac{1+\sqrt{3}i}{2}$, $a^4 = \dfrac{-1+\sqrt{3}i}{2}$, $a^6 = -1$, it follows that $a^{100} = (a^6)^{16}a^4 = \dfrac{-1+\sqrt{3}i}{2}$.

(4) Find $\cos\dfrac{\pi}{8}$.

(SOL) : Note that $\cos^2\dfrac{\pi}{8} = \dfrac{\cos\dfrac{\pi}{4}+1}{2} = \dfrac{\dfrac{\sqrt{2}}{2}+1}{2} = \dfrac{2+\sqrt{2}}{4}$. Since $\cos\dfrac{\pi}{8} > 0$, it follows that $\cos\dfrac{\pi}{8} = \dfrac{\sqrt{2+\sqrt{2}}}{2}$.

(5) When $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{100} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, find $a+b+c+d$.

(SOL) : Since $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{n} = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$, it follows that $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{100} = \begin{pmatrix} 1 & 200 \\ 0 & 1 \end{pmatrix}$.

Hence, $a+b+c+d = 202$.

(6) When an arithmetic sequence $\{a_n\}_{n=1}^{\infty}$ satisfies $a_1 + a_3 = 12,\ a_7 + a_9 = 34$, find a_{13} . Here an arithmetic sequence means a sequence $\{a_n\}_{n=1}^{\infty}$ such that $a_n - a_{n-1}$ is constant for all n .

(SOL) : Note that $a_n = a_1 + (n-1)d$, where d is the common difference.

Since $2a_1 + 2d = 12$ and $2a_1 + 14d = 34$, we get $d = \frac{11}{6}$ and $a_1 = \frac{25}{6}$.

Hence, $a_{13} = \frac{25}{6} + 12 \times \frac{11}{6} = \frac{157}{6}$.

(7) Find $\lim_{\theta \to 0} \frac{1-\cos\theta}{\sin\theta\ tg\theta}$.

(SOL) : Note that

$$\lim_{\theta \to 0} \frac{1-\cos\theta}{\sin\theta\ tg\theta} = \lim_{\theta \to 0} \frac{(1-\cos\theta)\cos\theta}{\sin\theta\ \sin\theta} = \lim_{\theta \to 0} \frac{(1-\cos\theta)(1+\cos\theta)\cos\theta}{\sin^2\theta\ (1+\cos\theta)}$$

$$= \lim_{\theta \to 0} \frac{\sin^2\theta\cos\theta}{\sin^2\theta\ (1+\cos\theta)} = \lim_{\theta \to 0} \frac{\cos\theta}{1+\cos\theta} = \frac{1}{2} .$$

(8) When $f(x) = \frac{(2x^2-3x+2)^{50}}{x^2+1}$, find $f'(1)$.

(SOL) : Since

$f'(x) = \frac{50(2x^2-3x+2)^{49}(4x-3)(x^2+1) - (2x^2-3x+2)^{50}(2x)}{(x^2+1)^2}$, it follows that

$f'(1) = \frac{50 \times 2 - 2}{4} = \frac{49}{2}$.

(9) Find $\lim_{n\to\infty} \frac{1^4 + 2^4 + 3^4 + \cdots + n^4}{n^5}$.

(SOL) : Note that $\lim_{n\to\infty} \frac{1^4 + 2^4 + 3^4 + \cdots + n^4}{n^5} = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^4 \frac{1}{n} = \int_0^1 x^4\,dx = \left[\frac{x^5}{5}\right]_0^1 = \frac{1}{5}$.

(10) Find the area of the region enclosed by $y = x^2 - 3x + 1$ and $y = -x^2 - x + 5$.

(SOL) : From $x^2 - 3x + 1 = -x^2 - x + 5$, we have $2x^2 - 2x - 4 = 0$, which is $x^2 - x - 2 = 0$, or $(x-2)(x+1) = 0$. Hence, $x = -1, 2$. This means that two curves meet at $x = -1$ and $x = 2$. Since $-x^2 - x + 5 \geq x^2 - 3x + 1$ on $[-1, 2]$, the area of the region is

$$\int_{-1}^{2} \{(-x^2 - x + 5) - (x^2 - 3x + 1)\}\,dx = \int_{-1}^{2} (-2x^2 + 2x + 4)\,dx = \left[-\frac{2}{3}x^3 + x^2 + 4x\right]_{-1}^{2} = 9 .$$