2021 IUT Admission Test (SOCIE, Contract-Based)

(1) When $\alpha = \dfrac{5}{\sqrt{8}+\sqrt{3}}$ and $\beta = \dfrac{5}{\sqrt{8}-\sqrt{3}}$,

find $\alpha^3+\beta^3$.

(SOL) : Note that $\alpha = \dfrac{5}{\sqrt{8}+\sqrt{3}} = \dfrac{5(\sqrt{8}-\sqrt{3})}{5} = \sqrt{8}-\sqrt{3}$ and

$\beta = \dfrac{5}{\sqrt{8}-\sqrt{3}} = \dfrac{5(\sqrt{8}+\sqrt{3})}{5} = \sqrt{8}+\sqrt{3}$. Hence, $\alpha+\beta = 2\sqrt{8} = 4\sqrt{2}$, $\alpha\beta = 5$

and $\alpha^2+\beta^2 = 2(8+3) = 22$. It follows that

$$\alpha^3+\beta^3 = (\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2) = 4\sqrt{2}\times(22-5) = 68\sqrt{2} .$$

(2) When $\sin\alpha + \cos\alpha = \dfrac{1}{4}$ for $0 \le \alpha \le \dfrac{\pi}{4}$, find $\sin\alpha - \cos\alpha$.

(SOL) : Since $\dfrac{1}{16} = (\sin\alpha+\cos\alpha)^2 = 1+2\sin\alpha\cos\alpha$, we get $2\sin\alpha\cos\alpha = -\dfrac{15}{16}$.

Note that $(\sin\alpha-\cos\alpha)^2 = 1-2\sin\alpha\cos\alpha = 1+\dfrac{15}{16} = \dfrac{31}{16}$.

Since $\sin\alpha-\cos\alpha \le 0$ for $0 \le \alpha \le \pi/4$, it follows that $\sin\alpha-\cos\alpha = -\dfrac{\sqrt{31}}{4}$.

(3) Evaluate $\displaystyle\sum_{n=1}^{100}\left(\frac{1+i}{1-i}\right)^n$.

(SOL) : We note that $\dfrac{1+i}{1-i} = \dfrac{(1+i)(1+i)}{(1-i)(1+i)} = \dfrac{2i}{2} = i$.

$$\sum_{n=1}^{100}\left(\frac{1+i}{1-i}\right)^n = \sum_{n=1}^{100} i^n = \frac{i^{101}-i}{i-1} = \frac{i(i^{100}-1)}{i-1} = 0 .$$

(4) Find the sum of all integers a such that $3^{2x}-3^{x+1}+a=0$ has two distinct real solutions.

(SOL) : The equation is $(3^x)^2-3(3^x)+a=0$, which yields $3^x = \dfrac{3\pm\sqrt{9-4a}}{2}$.

Hence, when $9-4a>0$ and $3-\sqrt{9-4a}>0$, the equation has two distinct real solutions. From $9-4a>0$, we get $a<\dfrac{9}{4}$. From $3-\sqrt{9-4a}>0$, we get $\sqrt{9-a} < 3$, which is $9-a<9$ and $a>0$. Hence, $0<a<\dfrac{9}{4}$ and the possible integers between 0 and $\dfrac{9}{4}$ are 1, 2, whose sum is 3.

(5) When $f(x) = \dfrac{\sin(x^2)}{x}$, find $f'(\sqrt{\pi})$.

(SOL) : Since $f'(x) = \dfrac{2x^2\cos(x^2) - \sin(x^2)}{x^2}$, it follows that

$$f'(\sqrt{\pi}) = \frac{2\pi\cos(\pi) - \sin(\pi)}{\pi} = -2 .$$

(6) Evaluate $\displaystyle\int_1^2 x^3\sqrt{x^2-1}\, dx$.

(SOL) : Putting $t = x^2 - 1$, then $x^2 = t+1$ and $x\,dx = \dfrac{1}{2}dt$.

$$\int_1^2 x^3\sqrt{x^2-1}\, dx = \int_1^2 x^2\sqrt{x^2-1}\, x\,dx = \frac{1}{2}\int_0^3 (t+1)t^{1/2}dt = \frac{1}{2}\int_0^3 (t^{3/2}+t^{1/2})dt$$

$$= \frac{1}{2}\left[\frac{2}{5}t^{5/2} + \frac{2}{3}t^{3/2}\right]_0^3 = \frac{\sqrt{3}}{2}\left(\frac{18}{5}+2\right) = \frac{14\sqrt{3}}{5} .$$

(7) Find the area of the region enclosed by $y = x^2 + x$ and $y = -x^2 + 3x$.

(SOL) : Since two curves $y = x^2 + x$ and $y = -x^2 + 3x$ intersect at $x = 0, 1$, the area of the region is

$$\int_0^1 [(-x^2+3x) - (x^2+x)]dx = \int_0^1 (-2x^2+2x)dx = \left[-\frac{2}{3}x^3 + x^2\right]_0^1 = \frac{1}{3} .$$