2021 IUT Admission Test (SOCIE Scholarship)

(1) When $\alpha^2 = 4 + \sqrt{12}$ and $\beta^2 = 4 - \sqrt{12}$, find $\alpha^3 + \beta^3$, where $\alpha > 0$ and $\beta > 0$.

(SOL) : Since $\alpha > 0$, it follows that
$\alpha = \sqrt{4 + \sqrt{12}} = \sqrt{4 + 2\sqrt{3}} = \sqrt{(\sqrt{3} + 1)^2} = \sqrt{3} + 1$. Similarly, since $\beta > 0$, we get $\beta = \sqrt{4 - \sqrt{12}} = \sqrt{4 - 2\sqrt{3}} = \sqrt{(\sqrt{3} - 1)^2} = \sqrt{3} - 1$. Hence, $\alpha + \beta = 2\sqrt{3}$ and $\alpha\beta = 2$. Since $\alpha^2 + \beta^2 = 8$, it follows that

$$\alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2) = 2\sqrt{3}(8 - 2) = 12\sqrt{3} .$$

(2) Evaluate $tg\frac{3\pi}{10}\, tg\frac{6\pi}{5}$.

(SOL) : We note that

$$tg\frac{3\pi}{10}\, tg\frac{6\pi}{5} = tg\frac{3\pi}{10}\, tg\frac{\pi}{5} = \frac{\sin\frac{3\pi}{10}}{\cos\frac{3\pi}{10}}\frac{\sin\frac{\pi}{5}}{\cos\frac{\pi}{5}} = \frac{\sin\left(\frac{\pi}{2} - \frac{\pi}{5}\right)}{\cos\left(\frac{\pi}{2} - \frac{\pi}{5}\right)}\frac{\sin\frac{\pi}{5}}{\cos\frac{\pi}{5}} = \frac{\cos\frac{\pi}{5}}{\sin\frac{\pi}{5}}\frac{\sin\frac{\pi}{5}}{\cos\frac{\pi}{5}} = 1 .$$

(3) Evaluate $\lim_{x \to 1} \frac{-2 + \sqrt{5 - x}}{-1 + \sqrt{2 - x}}$.

(SOL) : We note that

$$\lim_{x \to 1} \frac{-2 + \sqrt{5 - x}}{-1 + \sqrt{2 - x}} = \lim_{x \to 1} \frac{(\sqrt{5 - x} - 2)(\sqrt{5 - x} + 2)(\sqrt{2 - x} + 1)}{(\sqrt{2 - x} - 1)(\sqrt{2 - x} + 1)(\sqrt{5 - x} + 2)}$$

$$= \lim_{x \to 1} \frac{(5 - x - 4)(\sqrt{2 - x} + 1)}{(2 - x - 1)(\sqrt{5 - x} + 2)} = \frac{2}{4} = \frac{1}{2} .$$

(4) Find the maximum and minimum values of $f(x) = x^3 - 6x^2 + 9x + 4$ for $0 \le x \le 5$.

(SOL) : Since $f'(x) = 3x^2 - 12x + 9 = 3(x - 1)(x - 3)$, it follows that $f'(x) = 0$ at $x = 1$ and $x = 3$. Since $f(0) = 4$, $f(1) = 8$, $f(3) = 4$, $f(5) = 24$, the minimum value is 4 and the maximum value is 24 .

(5) Evaluate $\int_0^{\pi} x^2 \sin x \, dx$.

(SOL) : Using integration by parts, it follows that

$$\int x^2 \sin x \, dx = -x^2 \cos x + \int 2x \cos x \, dx = -x^2 \cos x + 2x \sin x - \int 2 \sin x \, dx$$

$$= -x^2 \cos x + 2x \sin x + 2\cos x + C,$$

which yields

$$\int_0^{\pi} x^2 \sin x \, dx = \left[-x^2 \cos x + 2x \sin x + 2\cos x\right]_0^{\pi} = \pi^2 - 4 .$$

(6) Find the volume of the solid obtained by rotating the region enclosed by $y = x^2$ and $y = \sqrt{8x}$ about the x-axis.

(SOL) : From $x^2 = \sqrt{8x}$, we get $x^4 = 8x$, which is $x(x^3 - 8) = x(x-2)(x^2+2x+4)$. Hence, two curves meet at $x = 0$ and $x = 2$. Hence, the volume is

$$\int_0^2 \pi\{(\sqrt{8x})^2 - (x^2)^2\} \, dx = \pi \int_0^2 (8x - x^4) \, dx = \pi \left[4x^2 - \frac{1}{5}x^5\right]_0^2 = \pi\left(16 - \frac{32}{5}\right) = \frac{48\pi}{5} .$$