The Solution of SOCIE Scholar Sample Test

1. Find $\lim_{x \to 0} \dfrac{\log_6(x^2+x+1)}{2^x+3^x-2}$.

Sol) Setting $f(x) = \log_6(x^2+x+1)$ and $g(x) = 2^x+3^x-2$, then $f(0) = g(0) = 0$. We note that

$$\lim_{x \to 0} \frac{\log_6(x^2+x+1)}{2^x+3^x-2} = \lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f(x)-f(0)}{g(x)-g(0)} = \lim_{x \to 0} \frac{\dfrac{f(x)-f(0)}{x-0}}{\dfrac{g(x)-g(0)}{x-0}} = \frac{f'(0)}{g'(0)} .$$

Since $f'(x) = \dfrac{1}{\ln 6} \dfrac{2x+1}{x^2+x+1}$ and $g'(x) = 2^x \ln 2 + 3^x \ln 3$, it follows that

$$\lim_{x \to 0} \frac{\log_6(x^2+x+1)}{2^x+3^x-2} = \frac{f'(0)}{g'(0)} = \frac{\dfrac{1}{\ln 6}}{\ln 2 + \ln 3} = \frac{1}{(\ln 6)^2} .$$

The answer is ⑤.

2. When t is a solution of $x^6+x^5+\cdots+x+1=0$, and $\sum_{n=0}^{50} t^n = at^2+bt+c$ for some integers a, b, c , find $a+b+c$.

Sol) Since $t^7-1=(t-1)(t^6+t^5+\cdots+t+1)=0$, it follows that $t^7=1$. Hence,

$$\sum_{n=0}^{50} t^n = \frac{1-t^{51}}{1-t} = \frac{1-(t^7)^7t^2}{1-t} = \frac{1-t^2}{1-t} = 1+t.$$ Hence, $a+b+c=0+1+1=2$.

The answer is ②.

3. When $A = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 3 \\ -2 & 7 \end{pmatrix}$ and $B^{-1}A^{27}B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, find $a+b+c+d$.

Sol) Simple computation shows that $A^2 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$, $A^3 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

Hence, $A^{27} = (A^3)^9 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, and

$B^{-1}A^{27}B = -B^{-1}B = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Thus, $a+b+c+d = -2$.

The answer is ①.

4. Find the sum of all solutions of

$$\sin x - \sqrt{3}\cos x = 1 \ , \ (0 \le x \le 2\pi).$$

Sol) We note that $\sin x - \sqrt{3}\cos x = 1$ is equal to $\frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x = \frac{1}{2}$, which is $\sin\left(x - \frac{\pi}{3}\right) = \frac{1}{2}$. Hence, $x - \frac{\pi}{3} = \frac{\pi}{6}, \frac{5\pi}{6}$. This gives $x = \frac{\pi}{2}, \frac{7\pi}{6}$. The sum is $\frac{\pi}{2} + \frac{7\pi}{6} = \frac{10\pi}{6} = \frac{5\pi}{3}$.

The answer is ⑤.

5. Find the maximum value of $f(x) = \frac{\sqrt{x}}{3x^2+1}$ for $x > 0$.

Sol) $$f'(x) = \frac{\frac{1}{2\sqrt{x}}(3x^2+1) - \sqrt{x}\,(6x)}{(3x^2+1)^2} = \frac{(3x^2+1) - 12x^2}{2\sqrt{x}\,(3x^2+1)^2} = \frac{1-9x^2}{2\sqrt{x}\,(3x^2+1)^2}$$

Hence, $f(x)$ has a maximum value at $x = \frac{1}{3}$, which is $f\left(\frac{1}{3}\right) = \frac{\sqrt{3}}{4}$.

The answer is ③.

6. Find the volume of the solid obtained by rotating the region enclosed by $y = -x^2 + 4$ and $y = x + 2$ about the x-axis.

Sol) Two graphs meet at $x = -2$ and $x = 1$. Hence the volume is given by

$$\pi\int_{-2}^{1}((-x^2+4)^2 - (x+2)^2)\,dx = \pi\int_{-2}^{1}(x^4 - 8x^2 + 16 - x^2 - 4x - 4)\,dx$$

$$= \pi\int_{-2}^{1}(x^4 - 9x^2 - 4x + 12)\,dx = \pi\left[\frac{1}{5}x^5 - 3x^3 - 2x^2 + 12x\right]_{-2}^{1}$$

$$= \pi\left(\frac{1}{5} - 3 - 2 + 12 - (-\frac{32}{5} + 24 - 7 - 24)\right) = \frac{108}{5}\pi \ .$$

The answer is ④.

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