2021 IUT Admission Test (SBL, Type A)

Math Examination

<Essay Types> Applicants should write detailed solving process. If there is no solution, you will receive 0 points regardless of the correct answer.

○ The point for each question is indicated next to each question number.

1. [5 points]

When $\sqrt{\dfrac{\sqrt[3]{3}}{\sqrt[4]{3}}}$ is written by $\sqrt[a]{b}$, find $a+b$.

2. [5 points]

Evaluate $\displaystyle\int_0^1 x(x^2+1)^9 dx$.

3. [5 points]

Find $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+1}+(n+1)\sqrt{n}}$.

4. [5 points]

When $f'(1)=3$, find $\displaystyle\lim_{x\to 1}\frac{f(x^2)-f(1)}{x-1}$.

5. [10 points]

Find the polynomial $f(x)$ satisfying

$$\lim_{x\to\infty}\frac{f(x)}{x^2+x+1}=2,\quad \lim_{x\to 1}\frac{f(x)}{x^2-4x+3}=4.$$

6. [10 points]

When α, β are solutions of $x^{\log_2 x} = 4x^2$, find $\alpha\beta$.

7. [10 points]

When $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$,

find $A^{-1}(3A+2B)B^{-1}$.

8. [10 points]

Find the maximum and minimum values of

$f(x) = \dfrac{2x^2+3x+2}{x^2+1}$.

9. [20 points]

Find the volume of the solid obtained by revolving the region enclosed by $y = \sqrt{x}$ and $y = \dfrac{1}{2}x$ about the x-axis.

10. [20 points]

When $g(x)$ is the inverse function of $f(x) = x^3 + x + 1$, evaluate

$\displaystyle\int_0^3 f(x)dx + \int_{f(0)}^{f(3)} g(x)dx$.

2021 IUT Admission Test Solution (SBL, Type A)

1. When $\sqrt{\dfrac{\sqrt[3]{3}}{\sqrt[4]{3}}}$ is written by $\sqrt[a]{b}$, find $a+b$.

sol) Since $\sqrt{\dfrac{\sqrt[3]{3}}{\sqrt[4]{3}}} = \dfrac{3^{1/6}}{3^{1/8}} = 3^{1/24} = \sqrt[24]{3}$, it follows that $a=24$, $b=3$ and $a+b=27$.

2. Evaluate $\int_0^1 x(x^2+1)^9 dx$.

sol) $\int_0^1 x(x^2+1)^9 dx = \left[\dfrac{1}{20}(1+x^2)^{10}\right]_0^1 = \dfrac{1023}{20}$.

3. Find $\sum_{n=1}^{\infty} \dfrac{1}{n\sqrt{n+1}+(n+1)\sqrt{n}}$.

sol) Observe that $\dfrac{1}{n\sqrt{n+1}+(n+1)\sqrt{n}} = \dfrac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n(n+1)}} = \dfrac{1}{\sqrt{n}} - \dfrac{1}{\sqrt{n+1}}$.

Then $\sum_{n=1}^{\infty} \dfrac{1}{n\sqrt{n+1}+(n+1)\sqrt{n}} = 1$.

4. When $f'(1)=3$, find $\lim_{x\to 1} \dfrac{f(x^2)-f(1)}{x-1}$.

sol) $\lim_{x\to 1} \dfrac{f(x^2)-f(1)}{x-1} = \lim_{x\to 1} \dfrac{f(x^2)-f(1)}{x^2-1} \times (x+1) = 2f'(1) = 6$.

5. Find the polynomial $f(x)$ satisfying
$\lim_{x\to\infty} \dfrac{f(x)}{x^2+x+1} = 2$, $\lim_{x\to 1} \dfrac{f(x)}{x^2-4x+3} = 4$.
sol) By the given conditions, we know that $f(x) = 2(x-1)(x+a)$.
Since $\lim_{x\to 1} \dfrac{f(x)}{x^2-4x+3} = \lim_{x\to 1} \dfrac{2(x-1)(x+a)}{(x-1)(x-3)} = -(1+a)$, we get $f(x) = 2(x-1)(x-5)$.

6. When α, β are solutions of $x^{\log_2 x} = 4x^2$, find $\alpha\beta$.
sol) Taking logarithm on both sides of the equation, we get
$(\log_2 x)^2 - 2\log_2 x - 2 = 0$. Since it has two roots α, β, we deduce that
$(\log_2 x)^2 - 2\log_2 x - 2 = (\log_2 x - \log_2 \alpha)(\log_2 x - \log_2 \beta)$.

Therefore $\log_2\alpha+\log_2\beta=2$, so that $\alpha\beta=4$.

7. When $A=\begin{pmatrix}1&2\\3&4\end{pmatrix}$, $B=\begin{pmatrix}1&-1\\2&1\end{pmatrix}$, find $A^{-1}(3A+2B)B^{-1}$.

sol) Since $A^{-1}=-\dfrac{1}{2}\begin{pmatrix}4&-2\\-3&1\end{pmatrix}$, $B^{-1}=\dfrac{1}{3}\begin{pmatrix}1&1\\-2&1\end{pmatrix}$, we have

$A^{-1}(3A+2B)B^{-1}=3B^{-1}+2A^{-1}=\begin{pmatrix}-3&3\\1&0\end{pmatrix}$.

8. Find the maximum and minimum values of $f(x)=\dfrac{2x^2+3x+2}{x^2+1}$.

sol) Since $f'(x)=\dfrac{3(x^2+1)-3x\cdot 2x}{x^2+1}=-\dfrac{3(x-1)(x+1)}{x^2+1}$, $f(x)$ is increasing for $-1<x<1$ and decreasing otherwise. We also observe $\lim\limits_{x\to\pm\infty}f(x)=2$. Therefore $f(x)$ has the maximum $\dfrac{7}{2}$ at $x=1$ and the minimum $\dfrac{1}{2}$ at $x=-1$.

9. Find the volume of the solid obtained by rotating the region enclosed by $y=\sqrt{x}$ and $y=\dfrac{1}{2}x$ about the x-axis.

sol) Two curves $y=\sqrt{x}$ and $y=\dfrac{1}{2}x$ meet at $x=0,4$. Since $0\le\dfrac{x}{2}\le\sqrt{x}$ for $0\le x\le 4$, the volume is given by

$$\int_0^4\pi\left\{(\sqrt{x})^2-\left(\frac{x}{2}\right)^2\right\}dx=\pi\int_0^4\left(x-\frac{x^2}{4}\right)dx=\pi\left[\frac{x^2}{2}-\frac{x^3}{12}\right]_0^4=\frac{8\pi}{3}.$$

10. When $g(x)$ is the inverse function of $f(x)=x^3+x+1$, evaluate $\displaystyle\int_0^3 f(x)dx+\int_{f(0)}^{f(3)}g(x)dx$.

sol) Observe that $\displaystyle\int_{f(0)}^{f(3)}g(x)dx=\int_0^3 g(f(t))f'(t)dt=\int_0^3 tf'(t)dt=[tf(t)]_0^3-\int_0^3 f(t)dt$.

Therefore, $\displaystyle\int_0^3 f(x)dx+\int_{f(0)}^{f(3)}g(x)dx=3f(3)=93$.

2021 IUT Admission Test(SOCIE, Contract-Based, Type A)

Math Examination

<Essay Types> Applicants should write detailed solving process. If there is no solution, you will receive 0 points regardless of the correct answer.

○ The point for each question is indicated next to each question number.

1. [5 points]
When α and β are solutions of $2x^2+5x+2=0$, find $\dfrac{\alpha}{\beta^2}+\dfrac{\beta}{\alpha^2}$.

2. [5 points]
Find the sum of all solutions of $\sqrt{3}\sin x+\cos x=\sqrt{2}$ for $0\le x\le 2\pi$.

3. [5 points]
Evaluate $\displaystyle\sum_{n=1}^{99}\left(\frac{1+i}{\sqrt{2}}\right)^{2n}$.

4. [5 points]
When $\log_a b=\dfrac{1}{2}$ and $2a+3b=20$ for $a>0$ and $b>0$, find ab.

5. [10 points]
Find $\displaystyle\lim_{x\to 0}\frac{\sqrt{x^2+4}-2}{x^3+2x^2}$.

6. [20 points]
Evaluate $\displaystyle\int_1^9 e^{2\sqrt{x}}\,dx$.

7. [20 points]
Find the area of the region enclosed by $y=x^4+3x^3-2x+1$ and $y=3x^3-2x+2$.

2021 IUT Admission Test (SOCIE, Contract-Based, Type A)

(1) When α and β are solutions of $2x^2+5x+2=0$, find $\dfrac{\alpha}{\beta^2}+\dfrac{\beta}{\alpha^2}$.

(SOL) : We note that $\alpha+\beta=-\dfrac{5}{2}$ and $\alpha\beta=1$. Hence,

$$\frac{\alpha}{\beta^2}+\frac{\beta}{\alpha^2} = \frac{\alpha^3+\beta^3}{\alpha^2\beta^2} = \frac{(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)}{(\alpha\beta)^2} = \frac{(\alpha+\beta)((\alpha+\beta)^2-3\alpha\beta)}{(\alpha\beta)^2}$$

$$= -\frac{5}{2}\times\left(\frac{25}{4}-3\right) = -\frac{5}{2}\times\frac{13}{4} = -\frac{65}{8}.$$

(2) Find the sum of all solutions $\sqrt{3}\sin x + \cos x = \sqrt{2}$ for $0 \le x \le 2\pi$.

(SOL) : From $\dfrac{\sqrt{3}}{2}\sin x + \dfrac{1}{2}\cos x = \dfrac{\sqrt{2}}{2}$, we have $\sin\left(x+\dfrac{\pi}{6}\right) = \dfrac{\sqrt{2}}{2}$.

Since $0 \le x \le 2\pi$, it follows that $x+\dfrac{\pi}{6} = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$. Hence, $x=\dfrac{\pi}{12}$ or $\dfrac{7\pi}{12}$.

The sum of two solutions is $\dfrac{\pi}{12}+\dfrac{7\pi}{12}=\dfrac{2\pi}{3}$.

(3) Evaluate $\displaystyle\sum_{n=1}^{99}\left(\frac{1+i}{\sqrt{2}}\right)^{2n}$.

(SOL) : Setting $\omega=\dfrac{1+i}{\sqrt{2}}$, it follows that $\omega^2=\left(\dfrac{1+i}{\sqrt{2}}\right)^2=\dfrac{2i}{2}=i$. Hence,

$$\sum_{n=1}^{99}\left(\frac{1+i}{\sqrt{2}}\right)^{2n} = \sum_{n=1}^{99} i^n = \frac{i(1-i^{99})}{1-i} = \frac{i(1-(-i))}{1-i} = \frac{i(1+i)}{1-i} = \frac{-1+i}{1-i} = -1.$$

(4) When $\log_a b=\dfrac{1}{2}$ and $2a+3b=20$ for $a>0$ and $b>0$, find ab.

(SOL) : Since $b=a^{\frac{1}{2}}$ or $a=b^2$, the equation $2a+3b=20$ is reduced to $2b^2+3b-20=0$. Since $(2b-5)(b+4)=0$ and $b>0$, it follows that $b=\dfrac{5}{2}$ and $a=b^2=\dfrac{25}{4}$. Hence, $ab=\dfrac{125}{8}$.

(5) Find $\lim_{x \to 0} \dfrac{\sqrt{x^2+4}-2}{x^3+2x^2}$.

(SOL) :
Note that

$$\lim_{x\to 0}\frac{\sqrt{x^2+4}-2}{x^3+2x^2} = \lim_{x\to 0}\frac{\left(\sqrt{x^2+4}-2\right)\left(\sqrt{x^2+4}+2\right)}{x^2(x+2)\left(\sqrt{x^2+4}+2\right)}$$

$$= \lim_{x\to 0}\frac{x^2+4-4}{x^2(x+2)\left(\sqrt{x^2+4}+2\right)} = \lim_{x\to 0}\frac{1}{(x+2)\left(\sqrt{x^2+4}+2\right)} = \frac{1}{8}\ .$$

(6) Evaluate $\int_1^9 e^{2\sqrt{x}}\,dx$.

(SOL) : Setting $t=\sqrt{x}$, then $dt = \dfrac{1}{2\sqrt{x}}dx = \dfrac{1}{2t}dx$ and hence $dx = 2t\,dt$.
It follows that

$$\int_1^9 e^{2\sqrt{x}}\,dx = \int_1^3 2t\,e^{2t}\,dt = \left[\,t\,e^{2t}\,\right]_1^3 - \int_1^3 e^{2t}dt = 3e^6 - e^2 - \left[\frac{1}{2}e^{2t}\right]_1^3$$

$$= 3e^6 - e^2 - \frac{1}{2}e^6 + \frac{1}{2}e^2 = \frac{5}{2}e^6 - \frac{1}{2}e^2\ .$$

(7) Find the area of the region enclosed by $y = x^4+3x^3-2x+1$ and $y=3x^3-2x+2$.

(SOL) : From the equation $x^4+3x^3-2x+1 = 3x^3-2x+2$, we have $x^4-1 = (x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1) = 0$ and $x=1,\ -1$. This means that two curves meet at $x=-1$ and $x=1$. Since $3x^3-2x+2 \geq x^4+3x^3-2x+1$ on $[-1,1]$, the area of the region is given by

$$\int_{-1}^1 \{(3x^3-2x+2)-(x^4+3x^3-2x+1)\}\,dx = \int_{-1}^1 (1-x^4)\,dx = 2\int_0^1 (1-x^4)\,dx$$

$$= 2\left[x - \frac{1}{5}x^5\right]_{x=0}^1 = \frac{8}{5}\ .$$

2021 IUT Admission Test(SOCIE)

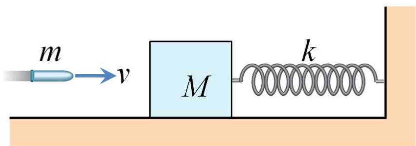
Physics Examination

<**Essay Types**> Applicants should write detailed solving process. If there is no solution, you will receive 0 points regardless of the correct answer.

○ The point for each question is indicated next to each question number.

○ Be sure to use SI units (the international system of units) for all physical quantities.

8. [10 points]
A block of mass $M = 5.0\,\mathrm{kg}$, at rest on a horizontal frictionless table, is attached to a rigid support by a spring of constant $k = 510\,\mathrm{N/m}$ as shown in the figure below. A bullet of mass $m = 0.1\,\mathrm{kg}$ and speed $v = 51\,\mathrm{m/s}$ strikes and is embedded in the block. Assuming the compression of the spring is negligible until the bullet is embedded, determine the maximum compression of the spring.



9. [10 points]
How much electrical energy is converted into heat in 1.0 minute by a 20-Ω resistor carrying 0.5 A of current?

10. [10 points]
Determine the minimum energy to ionize a hydrogen atom in the $n = 2$ state? Note that the energy of the ground state ($n = 1$) is -13.6 eV.

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2021 IUT Admission Test Answer Sheet for Contract-Based

1. [5 points]

Answer:

2. [5 points]

Answer:

3. [5 points]

Answer:

4. [5 points]

Answer:

5. [10 points]

Answer:

6. [20 points]

Answer:

7. [20 points]

Answer:

8. [10 points]

The total momentum of the system must be conserved before and after the collision.

$mv = (M+m)\,V$

Therefore, the speed of the block just after the collision

$V = \dfrac{m}{M+m}v = \dfrac{0.1\,\mathrm{kg}}{5.1\,\mathrm{kg}} \times 51\,\mathrm{m/s} = 1\,\mathrm{m/s}.$

Now due to the conservation of mechanical energy, the kinetic energy just after the collision should be same as the potential energy of the spring at the maximum compression. $\dfrac{1}{2}(M+m)\,V^2 = \dfrac{1}{2}kx^2$

Therefore,

$x = \sqrt{\dfrac{M+m}{k}}\,V = \sqrt{\dfrac{5.1\,\mathrm{kg}}{510\,\mathrm{N/m}}} \times 1\,\mathrm{m/s} = 0.1\,\mathrm{m}$

Answer: 0.1 m

9. [10 points]

The electrical energy converted into heat is given by $Q = I^2Rt$, where I is current, R is resistance, and t is time in seconds. Inserting the values in the problem, the heat is determined to be

$(0.5\,\mathrm{A})^2 \times 20\,\Omega \times 60\,\mathrm{sec} = 300\ \mathrm{J}$

Answer: 300 J

10. [10 points]

The energy level of a hydrogen atom is expressed as $E_n = \dfrac{-13.6\ \mathrm{eV}}{n^2}$, where n is an integer. So, the energy in the n = 2 state is $E_2 = \dfrac{-13.6\ \mathrm{eV}}{2^2} = -3.4\,\mathrm{eV}$.

Therefore, the minimum energy to ionize the hydrogen atom is $0 - E_2 = 3.4\ \mathrm{eV}$.

Answer: 3.4 eV

2021 IUT Admission Test(SBL, Type A)

Math Examination

<Essay Types> Applicants should write detailed solving process. If there is no solution, you will receive 0 points regardless of the correct answer.

○ The point for each question is indicated next to each question number.

1. [5 points]

1. Simplify $\log_2\left(1-\frac{1}{2}\right)+\log_2\left(1-\frac{1}{3}\right)+\log_2\left(1-\frac{1}{4}\right)$.

2. [5 points]

When $a_{10}=10$ and

$$a_{n+1}=a_n+3$$

for every integer n, find a_2.

3. [5 points]

When

$$\overline{AB}=6,\ \overline{BC}=7,\ \angle A=\frac{\pi}{3}$$

in a triangle ABC, find $\overline{AC}$.

4. [5 points]

Find $\lim_{x\to 0}\left(\frac{1}{x\sqrt{1+x}}-\frac{1}{x}\right)$.

5. [10 points]

Evaluate $\int_0^2 x\sqrt{x^2+1}\,dx$.

6. [10 points]

When $A=\begin{pmatrix}1 & 2\\ 3 & 1\end{pmatrix}$, $B=\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$,

find all real numbers k such that $A+kB$ is not invertible.

7. [10 points]

When α, β are solutions of

$$x^{\log_2 \frac{8}{x}} = \frac{1}{64}x^2,$$

find $\alpha\beta$.

8. [10 points]

Find the range of constant a such that

$$x^4-4x^3-2x^2+12x+a \geq 0$$

for every real number x.

9. [20 points]

When a polynomial $f(x)$ satisfies

$$f(x)+\int_0^x (x-t)f''(t)dt = x^3-x^2+5x+1$$

for every real number x, find $f(1)$.

10. [20 points]

Find the area of the region given by

$$\frac{5}{2}\,|\,x\,|\,-\frac{3}{2}x-3 \leq y \leq 9-x^2.$$

Solutions of 2021 IUT admission test(SBL, Type A)

1. Simplify $\log_2\left(1-\frac{1}{2}\right)+\log_2\left(1-\frac{1}{3}\right)+\log_2\left(1-\frac{1}{4}\right)$.

sol) $\log_2\left(1-\frac{1}{2}\right)+\log_2\left(1-\frac{1}{3}\right)+\log_2\left(1-\frac{1}{4}\right)$.

$=\log_2\left(\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\right)=\log_2 2^{-2}=-2$

2. When $a_{10}=10$ and $a_{n+1}=a_n+3$ for every integer n, find a_2.

sol) Since $a_9=a_{10}-3$, $a_8=a_9-3,\cdots$, we get $a_2=a_{10}-3\times 8=-14$.

3. When $\overline{AB}=6$, $\overline{BC}=7$, $\angle A=\frac{\pi}{3}$ in a triangle ABC, find $\overline{AC}$.

sol) Let $x=\overline{AC}$. By the cosine rule, $7^2=6^2+x^2-2\times 6\times x\times\frac{1}{2}$. Therefore, $x=3+\sqrt{22}$.

4. Find $\lim_{x\to 0}\left(\frac{1}{x\sqrt{1+x}}-\frac{1}{x}\right)$.

sol) $\lim_{x\to 0}\left(\frac{1}{x\sqrt{1+x}}-\frac{1}{x}\right)=\lim_{x\to 0}\frac{-1}{\sqrt{1+x}\,(1+\sqrt{1+x})}=-\frac{1}{2}$

5. Evaluate $\int_0^2 x\sqrt{x^2+1}\,dx$.

sol) By change of variable $u=x^2+1$, we get

$\int_0^2 x\sqrt{x^2+1}\,dx=\int_1^5\frac{1}{2}\sqrt{u}\,du=\left[\frac{1}{3}u^{3/2}\right]_1^5=\frac{5\sqrt{5}-1}{3}$.

6. When $A=\begin{pmatrix}1&2\\3&1\end{pmatrix}$, $B=\begin{pmatrix}1&-1\\1&1\end{pmatrix}$, find all numbers k such that $A+kB$ is not invertible.

sol) Since $A+kB=\begin{pmatrix}1+k&2-k\\3+k&1+k\end{pmatrix}$ is not invertible, we get $(1+k)^2-(2-k)(3+k)=0$ and hence $k=1,\ -\frac{5}{2}$.

7. When α, β are solutions of $x^{\log_2 \frac{8}{x}} = \frac{1}{64}x^2$, find $\alpha\beta$.

sol) Taking $\log_2$ on both sides of the equation, we get

$(3-\log_2 x)\log_2 x = -6+2\log_2 x$ and hence $(\log_2 x)^2 - \log_2 x - 6 = 0$.

Then it follows that $\log_2 x = 3, -2$ and $x = 2^3, 2^{-2}$. Therefore $\alpha\beta = 2$.

8. Find the range of constant a such that

$$x^4 - 4x^3 - 2x^2 + 12x + a \geq 0$$

for every real number x.

sol) Let $f(x) = x^4 - 4x^3 - 2x^2 + 12x$. Then $f'(x) = 4x^3 - 12x^2 - 4x + 12 = 4(x-3)(x-1)(x+1)$.

Thus f has local minimum -9 at $x = -1, 3$, and so $f(x) \geq -9$ for every real number x. Therefore $a \geq 9$.

9. When a polynomial $f(x)$ satisfies

$$f(x) + \int_0^x (x-t)f''(t)dt = x^3 - x^2 + 5x + 1$$

for every real number x, find $f(1)$.

sol) When $x = 0$, we get $f(0) = 1$. Differentiating the equation, we get

$f'(x) + \int_0^x f''(t)dt = 3x^2 - 2x + 5$ and hence $2f'(x) - f'(0) = 3x^2 - 2x + 5$.

Then $f'(0) = 5$ and $f'(x) = \frac{3}{2}x^2 - x + 5$.

Therefore $f(x) = \frac{1}{2}x^3 - \frac{1}{2}x^2 + 5x + 1$ and $f(1) = 6$.

10. Find the area of the region given by

$$\frac{5}{2}|x| - \frac{3}{2}x - 3 \leq y \leq 9 - x^2.$$

sol) Observe that the curve $y = \frac{5}{2}|x| - \frac{3}{2}x - 3 = \begin{cases} x-3 & \text{if } x \geq 0, \\ -4x-3 & \text{if } x < 0 \end{cases}$

meets $y = 9 - x^2$ at $x = -2, 3$. Therefore, the area is given by

$$\int_{-2}^0 \{(9-x^2)-(-4x-3)\}dx + \int_0^3 \{(9-x^2)-(x-3)\}dx = \frac{215}{6}.$$

2021 IUT Admission Test(SOCIE, Contracted-Based, Type A)

Math Examination

<Essay Types> Applicants should write detailed solving process. If there is no solution, you will receive 0 points regardless of the correct answer.

○ The point for each question is indicated next to each question number.

1. [5 points]

When $\alpha = \frac{1}{3}$, find $\frac{\sqrt{1+\alpha} - \sqrt{1-\alpha}}{\sqrt{1+\alpha} + \sqrt{1-\alpha}}$.

2. [5 points]

When $tg\,\alpha = \frac{1}{2}$ for $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$,

find $\cos 2\alpha$, where $tg\,\alpha = \frac{\sin\alpha}{\cos\alpha}$.

3. [5 points]

When $\left(\frac{\sqrt{2+2\sqrt{2}}}{1+\sqrt{2}+i}\right)^{12} = a+bi$ for some real numbers a and b , find $a+b$.

4. [5 points]

When $63^x = 16$ and $21^y = 32$, find $\left(\frac{1}{2}\right)^{\frac{5}{y}-\frac{4}{x}}$.

5. [10 points]

Find $\lim_{x\to 0} \frac{1-\cos(x^2)}{\sin^4 x}$.

6. [20 points]

Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x + x)^2\, dx$.

7. [20 points]

Find the area of the region enclosed by the following three curve and lines:

$y = x\ln x \ (x \geq 1)$, $x = e$ and the x-axis.

2021 IUT Admission Test (SOCIE, Contracted-Based, Type A)

(1) When $\alpha=\dfrac{1}{3}$, find $\dfrac{\sqrt{1+\alpha}-\sqrt{1-\alpha}}{\sqrt{1+\alpha}+\sqrt{1-\alpha}}$.

(SOL) : Note that $1+\alpha=\dfrac{4}{3}$ and $1-\alpha=\dfrac{2}{3}$.

$$\frac{\sqrt{1+\alpha}-\sqrt{1-\alpha}}{\sqrt{1+\alpha}+\sqrt{1-\alpha}}=\frac{\sqrt{\frac{4}{3}}-\sqrt{\frac{2}{3}}}{\sqrt{\frac{4}{3}}+\sqrt{\frac{2}{3}}}=\frac{\sqrt{4}-\sqrt{2}}{\sqrt{4}+\sqrt{2}}$$
$$=\frac{(\sqrt{4}-\sqrt{2})^2}{2}=3-2\sqrt{2}\ .$$

(2) When $tg\,\alpha=\dfrac{1}{2}$ for $-\dfrac{\pi}{2}<\alpha<\dfrac{\pi}{2}$, find $\cos 2\alpha$, where $tg\,\alpha=\dfrac{\sin\alpha}{\cos\alpha}$.

(SOL) : Since $tg\,\alpha=\dfrac{1}{2}>0$, it follows that $0<\alpha<\dfrac{\pi}{2}$.

By the trigonometric ratio, we get $\cos\alpha=\dfrac{2}{\sqrt{5}}$ and $\sin\alpha=\dfrac{1}{\sqrt{5}}$.

Hence, $\cos 2\alpha=\cos^2\alpha-\sin^2\alpha=\dfrac{4}{5}-\dfrac{1}{5}=\dfrac{3}{5}$.

(Alternative solution: Since $tg^2\alpha=\dfrac{1}{4}$, we get $\dfrac{1}{\cos^2\alpha}=\sec^2\alpha=1+tg^2\alpha=1+\dfrac{1}{4}=\dfrac{5}{4}$,

which yields $\cos^2\alpha=\dfrac{4}{5}$ and $\cos\alpha=\dfrac{2}{\sqrt{5}}$.

Hence, $\cos 2\alpha=2\cos^2\alpha-1=\dfrac{3}{5}$.

(3) When $\left(\dfrac{\sqrt{2+2\sqrt{2}}}{1+\sqrt{2}+i}\right)^{12}=a+bi$ for some real numbers a and b, find $a+b$.

(SOL) : Note that $(1+\sqrt{2}+i)^2=2+2\sqrt{2}+(2+2\sqrt{2})i=(2+2\sqrt{2})(1+i)$.

Putting $\alpha=\dfrac{\sqrt{2+2\sqrt{2}}}{1+\sqrt{2}+i}$, then $\alpha^2=\dfrac{1}{1+i}$ and $\alpha^4=\dfrac{1}{2i}$.

Hence, $\alpha^{12}=(\alpha^4)^3=\left(\dfrac{1}{2i}\right)^3=\left(-\dfrac{i}{2}\right)^3=\dfrac{1}{8}i$, which shows that $a=0,\ b=\dfrac{1}{8}$.

It follows that $a+b=\dfrac{1}{8}$.

(4) When $63^x=16$ and $21^y=32$, find $\left(\dfrac{1}{2}\right)^{\frac{5}{y}-\frac{4}{x}}$.

(SOL) : Since $63 = 16^{\frac{1}{x}} = 2^{\frac{4}{x}}$ and $21 = 32^{\frac{1}{y}} = 2^{\frac{5}{y}}$, we get

$$\left(\frac{1}{2}\right)^{\frac{5}{y} - \frac{4}{x}} = 2^{\frac{4}{x} - \frac{5}{y}} = 2^{\frac{4}{x}}\left(2^{\frac{5}{y}}\right)^{-1} = 63 \cdot 21^{-1} = \frac{63}{21} = 3.$$

(5) Find $\lim_{x \to 0} \frac{1 - \cos(x^2)}{\sin^4 x}$.

(SOL) : Note that $1 - \cos(x^2) = \frac{(1 - \cos(x^2))(1 + \cos(x^2))}{1 + \cos(x^2)} = \frac{\sin^2(x^2)}{1 + \cos(x^2)}$.

$$\lim_{x \to 0} \frac{1 - \cos(x^2)}{\sin^4 x} = \lim_{x \to 0} \left(\frac{\sin(x^2)}{x^2}\right)^2 \frac{1}{1 + \cos(x^2)} \frac{x^4}{\sin^4 x} = 1 \cdot \frac{1}{2} \cdot 1 = \frac{1}{2}.$$

(6) Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x + x)^2 \, dx$.

(SOL) : Note that $x \cos x$ is an odd function and $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \cos x \, dx = 0$.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x + x)^2 \, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos^2 x + x^2 + 2x \cos x) \, dx = 2 \int_0^{\frac{\pi}{2}} (\cos^2 x + x^2) \, dx \ .$$

We note that

$$\int_0^{\frac{\pi}{2}} \cos^2 x \, dx = \int_0^{\frac{\pi}{2}} \frac{1 + \cos 2x}{2} \, dx = \left[\frac{1}{2}\left(x + \frac{\sin 2x}{2}\right)\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4} \text{ and}$$

$$\int_0^{\frac{\pi}{2}} x^2 \, dx = \left[\frac{1}{3} x^3\right]_0^{\frac{\pi}{2}} = \frac{\pi^3}{24}.$$

Hence,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x + x)^2 \, dx = 2 \int_0^{\frac{\pi}{2}} (\cos^2 x + x^2) \, dx = 2\left(\frac{\pi}{4} + \frac{\pi^3}{24}\right) = \frac{\pi}{2} + \frac{\pi^3}{12} \ .$$

(7) Find the area of the region enclosed by the following three curve and lines: $y = x \ln x \ \ (x \geq 1)$, $x = e$ and the x-axis.

(SOL) : Note that the indicated region is $D = \{(x, y) \mid 1 \leq x \leq e, 0 \leq y \leq x \ln x\}$.
The area of D is

$$\begin{aligned} \int_1^e x \ln x \, dx &= \left[\frac{1}{2} x^2 \ln x\right]_1^e - \frac{1}{2} \int_1^e x \, dx \\ &= \frac{1}{2} e^2 - \left[\frac{1}{4} x^2\right]_1^e = \frac{1}{2} e^2 - \frac{1}{4}(e^2 - 1) \\ &= \frac{1}{4}(e^2 + 1) . \end{aligned}$$

2021 IUT Admission Test(SOCIE, Scholarship, Type A)

Math Examination

<Essay Types> Applicants should write detailed solving process. If there is no solution, you will receive 0 points regardless of the correct answer.

○ The point for each question is indicated next to each question number.

1. [5 points]
When $x^2 - x + 1 = 0$, find $\dfrac{x^{12}+3}{x^{2022}}$.

2. [5 points]
When $z = \dfrac{\sqrt{3}-i}{2}$ and $\dfrac{z^{10}+z^{-5}}{z^7 - z^{-2}} = a + bi$ for some real numbers a and b, find $a+b$.

3. [10 points]
When α and β are solutions to $\left(\dfrac{1}{x}\right)^{1-\log_2 x} = 64$, find $\alpha + \beta$.

4. [10 points]
When $f(x)$ is the distance between (x, x^2+2) and $\left(-2, \dfrac{5}{2}\right)$, find the minimum value of $f(x)$.

5. [20 points]
Evaluate $\displaystyle\int_0^{\pi} e^x \sin x \, dx$.

6. [20 points]
Find the volume of the solid obtained by rotating the region enclosed by $y = x^2$ and $y = 3x - 2$ about the x-axis.

2021 IUT Admission Test (SOCIE, Scholarship, Type A)

(1) When $x^2-x+1=0$, find $\dfrac{x^{12}+3}{x^{2022}}$.

(SOL) : Note that $(x+1)(x^2-x+1)=x^3+1=0$. Hence, we get $x^3=-1$, $x^6=1$. It follows that $\dfrac{x^{12}+3}{x^{2022}}=\dfrac{(x^6)^2+3}{(x^6)^{337}}=\dfrac{1+3}{1}=4$.

(2) When $z=\dfrac{\sqrt{3}-i}{2}$ and $\dfrac{z^{10}+z^{-5}}{z^7-z^{-2}}=a+bi$ for some real numbers a and b, find $a+b$.

(SOL) : Note that $z^2=\dfrac{2-2\sqrt{3}i}{4}=\dfrac{1-\sqrt{3}i}{2}$,

$z^4=\dfrac{-2-2\sqrt{3}i}{4}=\dfrac{-1-\sqrt{3}i}{2}=\dfrac{-i(\sqrt{3}-i)}{2}=-iz$, which shows that $z^3=-i$.

Hence, we get

$$\frac{z^{10}+z^{-5}}{z^7-z^{-2}}=\frac{z^{15}+1}{z^{12}-z^3}=\frac{-i+1}{1+i}=\frac{1-i}{1+i}=\frac{(1-i)^2}{(1+i)(1-i)}=\frac{-2i}{2}=-i.$$

It follow that $\dfrac{z^{10}+z^{-5}}{z^7-z^{-2}}=-i$, which yields $a=0$ and $b=-1$. Hence, $a+b=-1$.

(3) When α and β are solutions to $\left(\dfrac{1}{x}\right)^{1-\log_2 x}=64$, find $\alpha+\beta$.

(SOL) : Since $\left(\dfrac{1}{x}\right)^{1-\log_2 x}=64$, we get $(1-\log_2 x)\log_2\left(\dfrac{1}{x}\right)=\log_2 2^6$, which is $(\log_2 x)^2-(\log_2 x)-6=(\log_2 x-3)(\log_2 x+2)=0$. Hence, $x=2^3$ and $x=2^{-2}$. It follows that $\alpha+\beta=2^3+2^{-2}=\dfrac{33}{4}$.

(4) When $f(x)$ is the distance between (x, x^2+2) and $\left(-2, \dfrac{5}{2}\right)$, find the minimum value of $f(x)$.

(SOL) : Note that $f(x)=\sqrt{(x+2)^2+\left(x^2-\dfrac{1}{2}\right)^2}$ and

$f'(x)=\dfrac{2(x+2)+2\left(x^2-\dfrac{1}{2}\right)\cdot 2x}{2f(x)}=\dfrac{2(x^3+1)}{f(x)}=\dfrac{2(x+1)(x^2-x+1)}{2f(x)}$. Since $x^2-x+1>0$, $f(x)$ has a minimum value at $x=-1$, which is $f(-1)=\sqrt{1^2+\left(\dfrac{1}{2}\right)^2}=\dfrac{\sqrt{5}}{2}$.

(5) Evaluate $\int_0^{\pi} e^x \sin x\, dx$.

(SOL) : Using integration by parts, we have

$\int e^x \sin x\, dx = e^x \sin x - \int e^x \cos x\, dx = e^x \sin x - (e^x \cos x - \int e^x(-\sin x)\,dx)$, which yields

$\int e^x \sin x\, dx = e^x(\sin x - \cos x) - \int e^x \sin x\, dx$. Hence

$2\int e^x \sin x\, dx = e^x(\sin x - \cos x) + C$, and

$\int e^x \sin x\, dx = \frac{1}{2}e^x(\sin x - \cos x) + C$. Hence

$$\int_0^{\pi} e^x \sin x\, dx = \frac{1}{2}\left[e^x(\sin x - \cos x)\right]_0^{\pi} = \frac{1}{2}\{e^{\pi}(0+1) - e^0(0-1)\} = \frac{e^{\pi}+1}{2}.$$

(6) Find the volume of the solid obtained by rotating the region enclosed by $y = x^2$ and $y = 3x-2$ about the x-axis.

(SOL) : Since $x^2 - (3x-2) = x^2 - 3x + 2 = (x-1)(x-2) = 0$, two curves meet at $x=1$ and $x=2$. Since $(3x-2) \geq x^2 \geq 0$, the volume is given by

$$V = \pi\int_1^2 [(3x-2)^2 - (x^2)^2]\,dx = \pi\int_1^2 (9x^2 - 12x + 4 - x^4)\,dx = \pi\left[3x^3 - 6x^2 + 4x - \frac{x^5}{5}\right]_1^2 = \frac{4\pi}{5}.$$

2021 IUT Admission Test(SOCIE Scholar)

Physics Examination

<Essay Types> Applicants should write detailed solving process. If there is no solution, you will receive 0 points regardless of the correct answer.

○ The point for each question is indicated next to each question number.

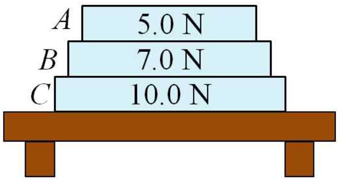
○ Be sure to use SI units (the international system of units) for all physical quantities.

1. [3 points]
There is a free-falling body initially at rest. If the distance moved by the body for the first 1 s is H, what is the distance moved by the body for the next 1 s?

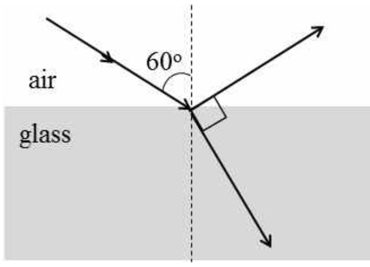
2. [3 points]
The charge on a parallel-plate capacitor is 6 × 10^{-6} C. If the distance between the plates is 3 mm and the capacitance is 4 μF, what is the strength of the electric field between the plates?

3. [3 points]
Determine the energy of a photon whose wavelength is 450 nm. (Assume that the Planck's constant is 6.6 × 10^{-34} J·s and the speed of light is 3.0 × 10^{8} m/s.)

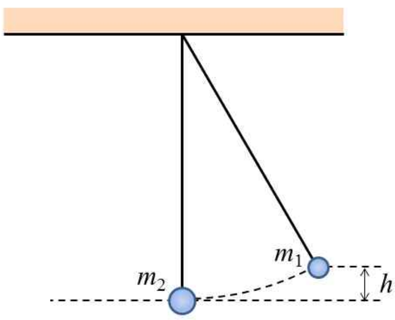
4. [6 points]
Three books are stacked on a table as shown in the figure. The weights of book A, B and C are 5.0 N, 7.0 N and 10.0 N, respectively. Determine the net force on book B.



5. [6 points]
A beam of light in air is incident on the smooth surface of a glass at an angle of incidence of 60° as shown below. If the reflected beam and refracted beam are perpendicular to each other, what is the index of refraction of the glass? (Note that the index of refraction of air is 1.0.)



6. [9 points]
Following figure shows two pendulums which consist of particles of masses $m_1 = 1.0\,\mathrm{kg}$ and $m_2 = 3.0\,\mathrm{kg}$ suspended from same fixed ends with massless strings of same length. The particle of mass m_1 is pulled aside to a hight $h = 8.0\,\mathrm{cm}$ from the lowest point and released initially at rest. The particle of mass m_2 is initially at rest at the lowest point. After the collision, two particles stick together and swing upward together. Find the maximum height that the stuck-together bodies can reach.



NAME:

2021 IUT Admission Test Answer Sheet for SOCIE Scholar Physics A

1. [3 points]

The distance moved by a free-falling body as a function of time t is $s(t) = \frac{1}{2}gt^2$. Therefore

$s(1\,\mathrm{s}) = \frac{1}{2}g \times 1\,\mathrm{s}^2 \equiv H$. $s(2\,\mathrm{s}) = \frac{1}{2}g \times 4\,\mathrm{s}^2 = 4H$. The distance moved by the body for the next 1 s is

$\Delta s = s(2\,\mathrm{s}) - s(1\,\mathrm{s}) = 4H - H = 3H$.

Answer: $3H$

2. [3 points]

The charge on a capacitor is given by $Q = CV$, where C is the capacitance and V is voltage.

Therefore, $V = \frac{Q}{C} = \frac{6 \times 10^{-6}\,\mathrm{C}}{4 \times 10^{-6}\,\mathrm{F}} = 1.5\,\mathrm{V}$.

Since the electric field is given by $E = \frac{V}{d}$, where d is the distance between the plates.

Consequently, we have $E = \frac{1.5\,\mathrm{V}}{3 \times 10^{-3}\,\mathrm{m}} = 500\,\mathrm{V/m}$

Answer: 500 V/m

3. [3 points]

The energy of a photon is given by $E = hf$, where h is Planck's constant and f is the frequency of the photon.

The frequency is given by $f = \frac{c}{\lambda}$, where c is the speed of light and f is the wavelength. By combining the equations,

$$E = \frac{hc}{\lambda} = \frac{(6.6 \times 10^{-34}\,\mathrm{J \cdot s})(3.0 \times 10^{8}\,\mathrm{m/s})}{4.5 \times 10^{-7}\,\mathrm{m}}$$
$$= 4.4 \times 10^{-19}\,\mathrm{J}$$

Answer: 4.4×10^{-19} **J**

4. [6 points]

There are three forces acting on book B: the downward gravitational force, the downward force from book A, and the upward force from book C.
The downward gravitational force = -7.0 N.
The downward force from book A = -5.0 N.
The upward force from book C = +5.0 N + 7.0 N = 12.0 N.
Therefore the net force on book B = -7.0 N − 5.0 N + 12.0 N = 0 N.
Alternatively, since the acceleration of book B is zero, the net force on book B is determined to be zero by using the Newton's 2nd law, $F_{net} = ma$.

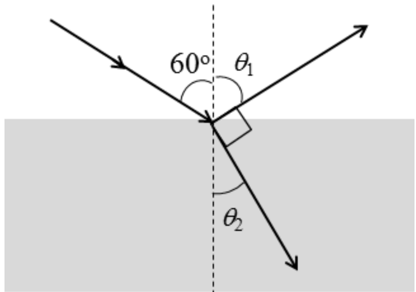
Answer: 0 N

5. [6 points]

The angle of reflection is the same as the angle of incidence. That is $\theta_1 = 60^\circ$. Since the reflected and refracted beams are perpendicular to each other, we have $\theta_2 = 30^\circ$. From Snell's law,

$n_a \sin 60^\circ = n_g \sin\theta_2 = n_g \sin 30^\circ$, where n_a and n_g are the index of refraction of air and glass, respectively.

Therefore, $\therefore n_g = \frac{\sin 60^\circ}{\sin 30^\circ} = \sqrt{3}$



Answer: $\sqrt{3}$

6. [9 points]

The speed of the particle of mass m_1 just before the collision (v_1) can be obtained by using

$m_1 gh = \frac{1}{2}m_1 v_1^2$. Therefore, $v_1 = \sqrt{2gh}$.

The speed of the stuck-together bodies just after the collision (V) can be obtained by using the conservation of linear momentum: $m_1 v_1 = (m_1 + m_2)V$. Therefore,

$$V = \frac{m_1}{m_1 + m_2}\sqrt{2gh}.$$

The maximum height that the stuck-together bodies can reach (H) is obtained by using the conservation of mechanical energy: $\frac{1}{2}(m_1 + m_2)V^2 = (m_1 + m_2)gH$.

Therefore, $H = \frac{V^2}{2g} = \frac{(\frac{m_1}{m_1 + m_2})^2 \times 2gh}{2g} = (\frac{m_1}{m_1 + m_2})^2 h$

$= (\frac{1.0\,\mathrm{kg}}{1.0\,\mathrm{kg} + 3.0\,\mathrm{kg}})^2 \times 8.0\,\mathrm{cm} = 0.5\,\mathrm{cm}$

Answer: 0.5 cm